Lecture 19 Summary

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Superconductors in a Magnetic Field - Vortex Lattice
The structure of a vortex lattice.

1 Order parameter solution

Last time we found the first nucleation field for superconductivity in a strong magnetic field for $T < T_c$ as
$H_{c2} = \sqrt{2}\kappa H_c$.
We took solutions to the linearized GL equation of the form,
$f(x, y) = g(x)e^{iky}$.
H_{c2} is the first non-trivial solution to the 1D differential equation:
$\frac{-\hbar^2}{2m^*}g'' + \frac{1}{2}k_s(x - x_0)^2 g = \frac{\hbar^2}{2m^*}\frac{g}{\xi_{GL}^2} = \epsilon g = -\alpha g$,
where the "spring constant" is $k_s \equiv \frac{(e^*\mu_0 H)^2}{m^*}$, and $x_0 \equiv \frac{\Phi_0 k}{2\pi\mu_0 H}$.

The eigenfunction for $g(x)$ is the ground state of the harmonic oscillator, with full solutions of the form,
$f(x, y; k) = e^{iky}e^{-(x-x_k)^2/2\xi_{GL}^2}$, with $x_k \equiv \frac{\Phi_0 k}{2\pi\mu_0 H}$.
This represents an infinite number of degenerate solutions, labeled by the parameter k.

We expect a set of solutions that are periodic in space. This can be accomplished by making k an integer multiple of a basic wavenumber q as $k = nq$, with $n = 0, \pm1, \pm2, ...$ Now the centers of the Gaussians are also periodic in space with $x_n = \frac{\Phi_0 nq}{2\pi\mu_0 H}$.
The y-solution is periodic with period $\Delta y = 2\pi/q$ and the x-solution has period $\Delta x = \frac{\Phi_0 q}{2\pi\mu_0 H}$. The area of a unit cell is therefore $\Delta x\Delta y = \frac{\Phi_0}{\mu_0 H}$, showing that exactly one flux quantum is confined in each unit cell. This is the ultimate limit imposed by quantum mechanics when the energy per unit area of an S/N interface is negative. The vortex lattice is the result of the proliferation of negative energy interfaces, arrested only by fluxoid quantization.

2 Vortex lattice solutions

The general solution for $\psi(x, y)$ comes from a linear superposition of all of the above solutions:
$\psi(x, y) = \sum_n C_n e^{inqy} e^{-(x-x_n)^2/2\xi_{GL}^2}$. If C_n is periodic in n then ψ is also periodic in space.
The two common solutions are the square lattice (obtained when $C_n = C_0$ for all n), and the triangular lattice (obtained when $C_1 = iC_0$ and $C_{n+2} = C_n$ for all n). The true minimum energy solution is found from GL theory by minimizing the free-energy difference,
$\langle f_s - f_n \rangle = -\frac{\alpha^2}{2\beta}\frac{1}{\beta_A}$, with
$\beta_A \equiv \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2}$. Hence we seek the minimum value of β_A. A uniform solution has $\beta_A = 1$. One finds that the square lattice has $\beta_A = 1.18$ while the triangular lattice has $\beta_A = 1.16$, just slightly lower. The class web site shows these solutions for $\psi(x, y)$ as well as many experimental techniques to image the vortex lattice. Some techniques (STM) measure the local density of states at the Fermi energy, which is enhanced in the vortex core due to the suppressed order parameter. Bitter decoration images the magnetic field concentration near the vortex cores. Lorentz microscopy magneto-optic imaging, SQUID and magnetic force microscopy methods all image the magnetic field profiles.

Note that we have assumed the superconductor does no screening, hence to this first approximation the magnetic field is homogeneous in the superconductor.

3 Estimation of H_{c1}

How do vortices first enter a type-II superconductor? As the applied field is increased from 0, the superconductor will maintain the Meissner state by creating screening currents to keep the flux out. It costs energy to create such screening currents. On the other hand, allowing the field to penetrate will force the superconductor to give up some condensation energy in the core of the vortex where the order parameter is suppressed (as seen in the $\psi(x, y)$ profiles discussed above). The first vortex will enter when these two energies are comparable. In other words when
$\frac{\mu_0 H_{c1}^2}{2}\pi\lambda_{eff}^2 L \approx \frac{\mu_0 H_c^2}{2}\pi\xi_{GL}^2 L$, where L is the length of the vortex in the superconductor. The left hand side is an estimate of the energy required to exclude a magnetic field of magntidue H_{c1} in a "'tube"' of radius λ_{eff}, while the right hand side is the condensation energy lost when the vortex core goes normal. The equality yields,
$H_c \approx \kappa H_{c1}$ with $\kappa = \lambda_{eff}/\xi_{GL}$. Hence $H_{c1} < H_c$ in type-II superconductors. Using our earlier result for the upper critical field, $H_{c2} = \sqrt{2}\kappa H_c$, one can find $H_c = \sqrt{H_{c1}H_{c2}/\sqrt{2}}$, the geometric mean.

4 Structure of an Isolated Vortex

We will attempt to solve the full nonlinear GL equation which is coupled to the current equation, self consistently. We make a number of assumptions based on the vortex lattice solution:
1. The magnetic field is in the z direction, $\overrightarrow{H} = H\widehat{z}$. We shall ignore any variation of the solution in the z direction (i.e. $\partial/\partial z$, etc. will be ignored).
2. The magnetic field is supported by currents that flow in the $x-y$ plane. Hence both $\overrightarrow{J}$ and $\overrightarrow{A}$ will be confined to that plane.
3. The solution will have full cylindrical symmetry. The order parameter will be of the form $\psi(\overrightarrow{r}) = \psi_\infty f(r)e^{i\theta}$. The explicit θ dependence is intentional. It gives rise to a phase pickup of 2π upon moving the wavefunction in a closed loop around the vortex core in a plane perpendicular to the magnetic field direction.
4. With this cylindrical symmetry, the vector potential is constrained to have only a radial dependence and a $\widehat{\theta}$ direction: $\overrightarrow{A}(\overrightarrow{r}) = A(r)\widehat{\theta}$.

With these observations, the coupled GL and current equations become:
$f - f^3 - \xi_{GL}^2\left[\left(\frac{1}{r} - \frac{2\pi}{\Phi_0}A(r)\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = 0$, where $f = \psi/\psi_\infty$ and the current density expression is
$\overrightarrow{J} = \frac{e^*}{m^*}\psi_\infty^2 f^2(r)\left[\frac{\hbar}{r}\widehat{\theta} - e^*\overrightarrow{A}(r)\widehat{\theta}\right]$ with
$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A} = \widehat{z}\frac{1}{r}\frac{\partial}{\partial r}(rA(r))$.

One can solve for the vector potential in terms of the magnetic field by integrating the last expression,
$A(r) = \frac{1}{r}\int_0^r \mu_0 h(r')r'dr'$, where $h(r)$ is the microscopic magnetic field.

Now examine the vector potential at small r (near the vortex core) and at large r, in sequence:
1. As $r \to 0$, on the scale of the GL coherence length, we expect that $h(r)$ is uniform since it is screened out on the much larger length scale λ_{eff} in the type-II high-κ limit. Hence $h(r) \approx (0)$ and we have,
$A(r \to 0) = \frac{\mu_0 h(0)}{2}r$.
2. As $r \to \infty$ the integral for $A(r)$ encompasses all of the flux in the vortex, which we know is Φ_0, and the result is $A(r \to \infty) = \frac{\Phi_0}{2\pi r}$.

Now examine the GL equation in each of these limits:
1. As $r \to 0$ we use the solution for $A(r)$ given above to find,
$f - f^3 - \xi_{GL}^2\left[\left(\frac{1}{r} - \frac{2\pi}{\Phi_0}\frac{\mu_0 h(0)}{2}r\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = 0.$
We try a power-law solution good near the origin, $f(r) = cr^n$. One finds that to leading order $n = 1$, which is exactly what we saw in the solution for $\psi(x, y)$ in the vortex lattice solution - ice cream cones. To next order one finds a correction to $f \propto r$ of cubic order in r/ξ_{GL}, with a minus sign. The solution can therefore be approximated as $f(r) \approx \tanh(r/\xi_{GL})$. This has the linear behavior at

small r and the asymptotic behavior $f = 1$ far from the vortex core, as expected.

Going back to the vector potential and magnetic field, we can utilize the fact that screening occurs on longer length scales than the core diameter ($\kappa >> 1$), hence $f(r) \approx 1$ for $r > \xi_{GL}$ in the current density equation above. With this, we can re-write the current density equation as,

$\mu_0 \lambda_{eff}^2 \overrightarrow{J} + A(r)\widehat{\theta} = \frac{\Phi_0}{2\pi r}\widehat{\theta}$

The left-hand side is what we formerly called the generalized London equation. Taking the time derivative gives the first London equation, while taking the curl gives the second London equation. In this case we have an inhomogeneous equation with a source term on the right hand side.

Taking the curl of both sides gives a second London equation with a vorticity source term:

$\mu_0 \lambda_{eff}^2 \overrightarrow{\nabla} \times \overrightarrow{J} + \overrightarrow{\nabla} \times (A(r)\widehat{\theta}) = \overrightarrow{\nabla} \times \left(\frac{\Phi_0}{2\pi r}\widehat{\theta} \right)$

The right hand side evaluates to a delta function at the origin: $\overrightarrow{\nabla} \times \left(\frac{\Phi_0}{2\pi r}\widehat{\theta} \right) = \Phi_0 \delta_2(r)\widehat{z}$, which we call the vorticity $\overrightarrow{V}(\overrightarrow{r})$. We can write the resulting equation as,

$\mu_0 \lambda_{eff}^2 \overrightarrow{\nabla} \times \overrightarrow{J} + \mu_0 \overrightarrow{h}(r) = \overrightarrow{V}(\overrightarrow{r})$.

To proceed with this 'mixed' equation, use the Maxwell equation $\overrightarrow{\nabla} \times \overrightarrow{h} = \overrightarrow{J}$ to get,

$\nabla^2 \overrightarrow{h} - \frac{1}{\lambda_{eff}^2} \overrightarrow{h} = -\frac{\Phi_0}{\mu_0 \lambda_{eff}^2} \delta_2(r)\widehat{z}$. This equation has an exact solution!